Industrial Automation

Reliability and Availability Exercises

Solutions

In our running example, we consider a chain of Christmas lights consisting of many light bulbs.

1. Let us first consider a single light bulb.
   (a) **Q:** Under what conditions can we assume that a single light bulb's failure rate is constant?
   A: We can assume that a light bulb's failure rate is constant if the life bulb is in the middle phase (also called "useful life") of its life cycle.
   (b) **Q:** From now on, let us assume that the failure rate of a single light bulb is indeed constant. Moreover, assume that a light bulb is expected to burn out after 1000 hours. What is its failure rate?
   A: The time after which the light bulb is expected to burn out is nothing else than it's mean time to failure (MTTF). Note that we also know that under the assumption of a constant failure rate λ, MTTF $= 1/\lambda$. Thus, $\lambda = 1/1000\ h^{-1}$
   (c) **Q:** What is the reliability of a single light bulb?
   A: Unlike the mean time to failure, reliability is a function of time. It expresses the probability of a system not having failed until a given time. For a constant failure rate λ, the reliability $R_B(t) = e^{-\lambda t} = e^{-t/1000}$. Note that the we consider hours as the units of t, as we express λ using the unit h^{-1}.
2. We consider a chain of Christmas lights consisting of 50 serially connected light bulbs. This means that if any light bulb burns out, the electric circuit is interrupted and all light bulbs go off.
   (a) **Q:** What is the reliability of the whole chain?
   A: As all components must work in order for the whole system to work, the reliability of the system is the product of the reliabilities of all its components. For our light chain, it is the product of reliabilities of each light bulb. Let $R_i(t)$ be the reliability of light bulb i, for $1 \leq i \leq 50$. As all light bulbs to have the same reliability $R_i(t) = R_B(t)$, the reliability of the whole (serially connected) chain is

$$R_S(t) = R_1(t) \cdot R_2(t) \cdots R_{50} = e^{-t/1000} \cdot e^{-t/1000} \cdots e^{-t/1000} = e^{-50t/1000} = e^{-t/20} \quad (1)$$

(b) **Q:** How long can we expect the light chain to work?

A: The reliability of the whole chain being $e^{-t/20}$, we see that the (constant) failure rate $\lambda = 1/20\ h^{-1}$. The time we expect the chain to work is the MTTF $= 1/\lambda = 20$ hours.

3. We now assume that the light bulbs are connected in parallel. Thus, if a light bulb burns out, the others may continue operating. We consider a light chain to be working as long as not more than 10 light bulbs are burned out.

(a) **Q:** What is the reliability of the whole chain?

A: Let R_B the reliability of a single light bulb. For simplicity of notation, we omit the time argument "(t)", but remember that the reliability is still a function of time!

The probability that exactly i out of 50 light bulbs fail is

$$\binom{50}{i} \cdot (1 - R_B)^i \cdot R_B^{50-i} \tag{2}$$

The part $(1 - R_B)^i$ signifies i particular bulbs failing while $(R_B)^{50-i}$ stands for the remaining bulbs not failing. The binomial coefficient represents the number of ways we can choose i out of 50 light bulbs.

If at most 10 light bulbs are allowed to fail, we need to calculate the probability that exactly zero of them fail, or exactly one fails, or exactly two fail ... or exactly 10 fail. Summing all of these probabilities gives us the resulting system reliability of he parallel version of the light chain:

$$R_P(t) = \sum_{i=0}^{10} \binom{50}{i} \cdot (1 - R_B)^i \cdot R_B^{50-i} = \sum_{i=0}^{10} \binom{50}{i} \cdot (1 - e^{-t/1000})^i \cdot (e^{-t/1000})^{50-i} \tag{3}$$

(b) **Q:** How long can we expect the light chain to work? (No need to calculate the value, just give a mathematical expression describing it.)

A: Again, we were asking for the mean time to failure. In general, the MTTF is calculated as the integral of the reliability function over time from 0 until infinity. Thus, in our case, the MTTF is

$$\int_0^\infty \sum_{i=0}^{10} \binom{50}{i} \cdot (1 - e^{-t/1000})^i \cdot (e^{-t/1000})^{50-i} dt \tag{4}$$

This evaluates to more than 245 hours (we did not ask you to calculate this), which is a substantial improvement over the serial version.

4. We assume that during the Christmas period, the light chain is used every evening for 4 hours during the whole month of December. Answer the following questions for both the serial and the parallel variant of the light chain.

(a) **Q:** What is the probability that the light chain will remain functional during the whole Christmas period? (Feel free to serve yourself with mathematical tools like WolframAlpha to compute the resulting numbers.)

A: In our definition, the Christmas period lasts $31 * 4 = 124$ hours. The probability that a system (in our case the light chain) will remain functional for a certain time t is expressed by its reliability. We only need to substitute t for 124 hours.

$$R_S(124) = e^{-124/20} \approx 0.2\% \tag{5}$$

$$R_P(124) = \sum_{i=0}^{10} \binom{50}{i} \cdot (1 - e^{-0.124})^i \cdot (e^{-0.124})^{50-i} \approx 97.3\% \tag{6}$$

Note that the serially connected light chain has extremely slim chance of surviving a Christmas period, while the vast majority of light chain survive (by our definition of "survive").

(b) **Q:** What is the probability that the light chain survives the Christmas periods of two consecutive years?

A: This means nothing else than doubling the number of hours the light chain needs to remain functional, i.e. computing $R_S(248)$ and $R_P(248)$.

$$R_S(248) = e^{-248/20} \approx 0.0004\% \tag{7}$$

$$R_P(248) = \sum_{i=0}^{10} \binom{50}{i} \cdot (1 - e^{-0.248})^i \cdot (e^{-0.248})^{50-i} \approx 44.72\% \tag{8}$$

The serially connected light chain's survival chances are almost zero, while almost half of the parallel chains survives two consecutive Christmas periods.

(c) **Q:** If, after the first year's Christmas period, 5 light bulbs burn out, what is the probability of the light chain surviving the next Christmas period? (Only relevant for the parallel case.)

A: This case is equivalent to considering a light chain consisting of 45 lights out of which only 5 are allowed to burn out, and evaluating its reliability (let us call it R'_P) for one Christmas period (of 124 hours).

$$R'_P(124) = \sum_{i=0}^{5} \binom{45}{i} \cdot (1 - e^{-0.124})^i \cdot (e^{-0.124})^{45-i} \approx 57\% \tag{9}$$

(d) **Q:** Assume that the package with the light chain contains 2 replacement light bulbs. How do the answers to the 3 above questions change?

A: Technically speaking, if "surviving" a Christmas means to continue functioning without any intervention, the answers do not change! The reliability of a system is a well defined term and does not take into account whether there are replacement parts available or not.

For replacement light bulbs to be useful, we need to make additional assumptions on repairing a (partially) broken light chain. If we assume that on the parallel version of the chain two light bulbs can be replaced before the whole chain fails, then we can model it as a system of 50 light bulbs out of which 12 (instead of 10) can fail.

For the serial version, we even need to redefine what "surviving" means. Otherwise, whenever a single light bulb fails, the chain is considered as having failed, regardless of the presence of 2 replacement light bulbs in the box. We can say, for example, that the serial version of the light chain fails if a light bulb goes of AND there is no more replacement light bulbs (the immediate use of which we assume for the first two failures). In this case, the probability of surviving a Christmas period is close to the case of a parallel version of the chain, where only 2 light bulbs are allowed to fail. It is however, not exactly the same probability (can you give an intuition why?).

(e) **Q:** What are the answers to the previous 4 questions if we use LED light bulbs with an expected life time of 5000 hours?

A: Here we use the very same formulas as above, but with different parameters, namely with the parameter to the exponential function being 5 times smaller (note that the MTTF of a single light bulb appears in the denominator). At the root of this change is the reliability of a single light bulb that changes from $e^{-t/1000}$ to $e^{-t/5000}$.

$$R_S(124) = e^{-124/100} \approx 28.94\% \tag{10}$$

$$R_P(124) = \sum_{i=0}^{10} \binom{50}{i} \cdot (1 - e^{-0.124/5})^i \cdot (e^{-0.124/5})^{50-i} \approx 99.999997\% \tag{11}$$

$$R_S(248) = e^{-248/100} \approx 8.37\% \tag{12}$$

$$R_P(248) = \sum_{i=0}^{10} \binom{50}{i} \cdot (1 - e^{-0.248/5})^i \cdot (e^{-0.248/5})^{50-i} \approx 99.998\% \tag{13}$$

$$R'_P(124) = \sum_{i=0}^{5} \binom{45}{i} \cdot (1 - e^{-0.124/5})^i \cdot (e^{-0.124/5})^{45-i} \approx 99.92\% \tag{14}$$

We see that using better components substantially helps improving the reliability of the chain in both the serial and the parallel case. While the serial version remains very bad nonetheless, the parallel version becomes extremely reliable (for a Christmas light chain - for other systems, e.g. safety-critical ones, even such high reliability might be insufficient in practice).

5. Now consider a case of a light chain that is supposed to be on permanently. If a light bulb goes off, it takes 1 hour to be noticed and replaced on expectation. For simplicity, we assume that repairs happen independently and can even be carried out in parallel.

(a) **Q:**What is the (stationary) availability of the serial variant of the light chain?

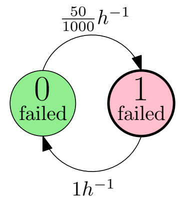
A:The availability of a system expresses what fraction of the time the system is in a working state. Thus, it is the time the system is working divided by the sum of the time it is working and the time it is being repaired, expressed mathematically, the availability is

$$\frac{MTTF}{MTTF + MTTR} = \frac{20}{20 + 1} \approx 95.24\% \tag{15}$$

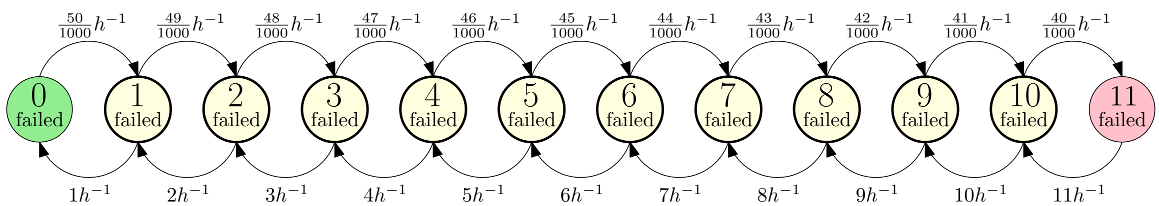
We substituted 20 for the MTTF, as we computed it in a previous exercise, and 1 for MTTR, as given in this question. We can say that, on average, the light chain works for 20 hours, then for one hour it is being repaired, then it works for 20 hours again, and so on...

(b) **Q:** Draw a Markov chain modeling the reliability of both the serial and the parallel variant of the light chain. For simplicity, assume that in the parallel case, the light chain goes off automatically when more than 10 light bulbs burn out.

A: For the serial case, the Markov chain only consists of two states, as there are only two possibilities for the light chain: either all light bulbs work or the whole chain is off after the first light bulb fails. The transition from the working state to the failed state happens with the light chain's failure rate $(\lambda) = 1/20\ h^{-1}$, which we have already before. The repair rate, which is, analogous to the failure rate, the inverse of the MTTR, is 1 h^{-1}



The parallel case is slightly more complex than the serial one, as we need to distinguish more states of the light chain, depending on how many light bulbs have failed. Also, the failure rates for each depend on the number of working light bulbs. A single light bulb having a failure rate of $1/1000\ h^{-1}$, the combined failure rate of n light bulbs is $n \cdot 1/1000\ h^{-1}$ (recall how we computed the reliability in question 2.(a)). The analogous is true for the repair rates.



(c) **Q:** Set up the differential equations describing the states of the Markov chains.

A: For each state p_i, we have one equation constructed using the all the incoming and outgoing edges of that state, of the form

$$\frac{dp_i}{dt} = \sum \lambda_k p_k(t) - \sum \lambda_i p_i(t) \tag{16}$$

where k goes through all the neighboring states from which there is an edge to p_i, while λ_i goes through all the outgoing edges of p_i.

Thus, the equations for the serial version are

$$\frac{dp_0(t)}{dt} = 1 \cdot p_1(t) - \frac{50}{1000} \cdot p_0(t) \tag{17}$$

$$\frac{dp_1(t)}{dt} = \frac{50}{1000} \cdot p_0(t) - 1 \cdot p_1(t) \tag{18}$$

and the equations for the parallel version are

$$\frac{dp_0(t)}{dt} = 1 \cdot p_1(t) - \frac{50}{1000} \cdot p_0(t) \tag{19}$$

$$\frac{dp_1(t)}{dt} = \frac{50}{1000} \cdot p_0(t) + 2 \cdot p_2(t) - \frac{49}{1000} \cdot p_1(t) - 1 \cdot p_1(t) \tag{20}$$

$$\frac{dp_2(t)}{dt} = \frac{49}{1000} \cdot p_1(t) + 3 \cdot p_3(t) - \frac{48}{1000} \cdot p_2(t) - 2 \cdot p_2(t) \tag{21}$$

$$\frac{dp_3(t)}{dt} = \frac{48}{1000} \cdot p_2(t) + 4 \cdot p_4(t) - \frac{47}{1000} \cdot p_3(t) - 3 \cdot p_3(t) \tag{22}$$

$$\cdots \tag{23}$$

$$\frac{dp_{10}(t)}{dt} = \frac{41}{1000} \cdot p_9(t) + 11 \cdot p_{11}(t) - \frac{40}{1000} \cdot p_{10}(t) - 10 \cdot p_{10}(t) \tag{24}$$

$$\frac{dp_{11}(t)}{dt} = \frac{40}{1000} \cdot p_{10}(t) - 11 \cdot p_{11}(t) \tag{25}$$

(d) **Q:** What are the initial conditions of the differential equations for a new light chain (i.e. when none of the light bulbs burned out yet)?

A: When the light chain is new, we assume all the light bulbs to be working. Thus, for both the serial and parallel version of the chain, at time $t = 0$, $p_0(0) = 1$ and $p_i(0) = 0$ for all $i > 0$.

(e) **Q:** How would you calculate the mean time to failure (MTTF) of the light chains? (Describe the steps, without performing the actual calculation.)

A: As described in class, to compute the MTTF of a system modeled by a Markov chain, we need to

i. Set up the differential equations.
ii. Identify the terminal (absorbing) states, i.e. the states with no outgoing edges.
iii. Solve the linear equation system using the Laplace transform.
iv. Compute the MTTF as the sum of the integrals over the functions corresponding to all non-terminal states.

(f) **Q:** How would you calculate the availability of the light chains? (Describe the steps, without performing the actual calculation.)

A: To compute the availability of a system, we need to:

i. Set up the differential equations according to the Markov chain.
ii. Identify all states in which the system is considered up (working) and down (failed).
iii. Remove one state equation (in principle an arbitrary one, but for numerical reasons, one corresponding to an unlikely state should be removed.)
iv. Add $1 = \sum p$ as the first equation.
v. Solve the linear equation system, yielding the fraction of time each state is visited.
vi. Compute the unavailability as the sum of all the down states.