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Quantum Optics 2 – Spring semester 2024 – 11/04/2024 and 18/04/2024
Problem Set 6 : Atom in Cavity

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This exercise sheet discusses the physics of an atom inside a cavity, a classical problem in quantum optics, in great detail. It is very long, but that is why you will be asked to solve it in two weeks time.

I. LINDBLAD MASTER EQUATION FOR A CAVITY WITH CLASSICAL DRIVE

The field inside a cavity can be described by the Hamiltonian $\mathcal{H}_C = \omega_c \hat{a}^\dagger \hat{a}$ with cavity frequency ω_c and cavity field annihilation operator $\hat{a}$. Note, we set $\hbar = 1$. An external **classical** drive can be described by the drive Hamiltonian $\mathcal{H}_d = \sqrt{\kappa}\mathcal{E}_0 \left(\hat{a}^\dagger e^{-i\omega t} - \hat{a} e^{i\omega t}\right)$ in the rotating wave approximation (RWA) with cavity decay rate κ.

1. Starting from the general drive $F\cos(\omega t + \phi) \cdot \hat{X}$ with frequency ω, phase ϕ and light amplitude quadrature $\hat{X} = 1/\sqrt{2}(\hat{a}^\dagger + \hat{a})$ write the driving in the interaction picture/rotating frame of the drive ω and apply the RWA by applying the right unitary transformation $\hat{U}$. You can use :

$$e^Z Y e^{-Z} = \sum_{n=0}^{\infty} \frac{[(Z)^n, Y]}{n!}, \quad \text{where,} \quad [(Z)^n, Y] \equiv [Z, [(Z)^{n-1}, Y]], \quad [(Z)^0, Y] \equiv Y$$

2. Rewrite the full Hamiltonian in the rotating frame of the drive as $\mathcal{H} = \Delta a^\dagger a + i\sqrt{\kappa}\mathcal{E}_0 \left(a^\dagger - a\right)$ with detuning $\Delta = \omega - \omega_c$.

3. The cavity can also decay through losses to the outside world. What is the Lindblad operator corresponding to cavity decay with a rate κ and the Lindblad master equation describing the time evolution of the density matrix of the cavity.

4. Using the master equation, derive the expression for time evolution of the expectation value of the annihilation operator : $\langle \dot{a} \rangle = \mathrm{Tr}(\dot{\rho} a) = -i\Delta \langle a \rangle - \frac{\kappa}{2} \langle a \rangle + \sqrt{\kappa}\mathcal{E}_0$.

5. What is the steady-solution $\langle a \rangle_{ss}$ of the cavity field? Which physical interpretation have the terms in the equation for $|\langle a \rangle_{ss}|^2$ in the case $\Delta = 0$. What is the dimension of $\mathcal{E}_0$?

II. JAYNES-CUMMINGS WITH DISSIPATION

Let's introduce a two-level system inside our cavity, that has an energy splitting ω_a and interacts with the field with a coupling strength g (equivalent to a Rabi frequency $\Omega = 2g$).

1. Starting from the quantized atom-cavity field interaction $g\hat{X}\hat{\sigma}_x$ write the interaction in the rotating frame of the drive and apply the RWA. What is the full Hamiltonian of atom and cavity in the rotation frame of the cavity drive? Define atom-drive detuning $\Delta_a = \omega - \omega_a$.

2. What is the Linblad operator for spontaneous emission of the atom, with a rate Γ and the corresponding Lindblad equation for the atom?

3. Derive the differential equation for the time evolution of the annihilation operator: $\langle \dot{a} \rangle = -(i\Delta + \kappa/2) \langle a \rangle + \sqrt{\kappa}\mathcal{E}_0 - ig \langle \hat{\sigma}_- \rangle$.

4. We now need to do the same for $\langle \dot{\sigma}_- \rangle$. In the expression, a term $\langle \hat{a}\hat{\sigma}_z \rangle$ should appear. Also compute the expression for $\langle \dot{\sigma}_z \rangle$.

We now have a set of three coupled differential equations, which are sufficient to describe the time evolution of the system, but require a lot of work to solve. Note that they include higher order correlations terms (expectation values of operators mixing atomic and cavity operations, such as $\langle a\sigma_z\rangle$ of $\langle a\sigma_+\rangle$). If we want an idea of the behavior and give up on the exact correlations, we can do a mean field approximation and factor these terms, as $\langle a\sigma_z\rangle \to \langle a\rangle \langle\sigma_z\rangle$. We can then write the corresponding steady-state equations and extract the important parameters for the problem.

5. Write the coupled steady-state equations of $\langle\hat{a}\rangle, \langle\hat{\sigma}_-\rangle, \langle\hat{\sigma}_z\rangle$ using the following expressions: $\delta_a = 2\Delta_a/\Gamma$, $\delta_c = 2\Delta/\kappa$, $\tilde{\Gamma} = \Gamma + 2i\Delta_{at}$ and $\tilde{\kappa} = \kappa + 2i\Delta$, photon number $n = |\langle a\rangle|^2$, the cooperativity $C = \frac{4g^2}{\kappa\Gamma}$ and the critical photon number $n_0 = \frac{\Gamma^2}{8g^2}$.

   By comparing your results for $\langle\sigma_z\rangle$ with the optical Bloch equations, give an interpretation for n_0.

6. Introduce the normalized intracavity intensity $I_c = n/n_0$, drive intensity $I_d = \frac{4|\mathcal{E}_0|^2}{n_0\kappa}$ and show that

$$I_d = I_c\left[\left(1 + \frac{C}{1 + I_c + \delta_{at}^2}\right)^2 + \left(\delta_c - \frac{C\delta_{at}}{1 + I_c + \delta_{at}^2}\right)^2\right]. \quad (1)$$

As already mentioned, the dimensionless quantity C, calculated at resonance, is called the cooperativity of the atom-cavity system. It quantifies how much the atom coherently exchanges an excitation with the cavity-mode as opposed to decaying into free space.

Let's now assume that we drive the cavity with a weak field, so that we stay in the low saturation regime, i.e $\langle\sigma_z\rangle \simeq -1$. For the rest of this section, we will also assume that $\Delta_a = \Delta_c$, i.e that the cavity is on resonance with the atomic transition.

7. What is the condition on n/n_0 to be in the low-saturation regime? Simplify the expressions in that case.

8. Simplify for the case $C \ll 1$. What is the cavity "spectrum" (I_c/I_d as a function of Δ_c)? What is the effect of the atom in that case?

9. Differentiate the low-saturation expression for $I_c(I_d)$ to find three local extrema. One of them is located at $\Delta = 0$. Check that for $g \gg \kappa, \Gamma$ (strong-coupling regime), $\Delta = -\sqrt{C\kappa\Gamma}/2$ and $\Delta = +\sqrt{C\kappa\Gamma}/2$ are also extrema.

The three extrema, that we found, are actually two maxima at $\Delta = \pm g$ and one minimum in $\Delta = 0$. The spectrum of the atom-cavity system now shows two peaks separated by $2g = \sqrt{C\kappa\Gamma}$.

10. What does this $2g$ energy gap remind you of? Give a condition on C for these two peaks to appear clearly in the spectrum.

11. Now, use Python to plot I_c/I_d as a function of Δ for different values of C, with fixed κ and Γ that are of the same order of magnitude. To help, you can express δ_c in terms of δ_{at},C and n_0. BE CAREFUL! You have to be a bit subtle to get the right normalizations...

This double-peak structure for a low-saturation atom-cavity system is called "Vacuum Rabi Splitting". Let's try to learn a bit more about it with Qutip !

III. STRONG COUPLING REGIME : VACUUM RABI OSCILLATIONS

Imagine we start with an atom in the excited state and an empty cavity. The state of the system can be written $|\psi\rangle = |e\rangle \otimes |n = 0\rangle$. If the "cooperativity" $C = \frac{4g^2}{\kappa\Gamma}$ is high enough, then the atom will decay from the excited state by emitting a photon in the cavity mode instead of in free space. This photon can then be reabsorbed and re-emitted a certain number of times, giving rise to "vacuum Rabi oscillations".

The term "strong coupling regime" corresponds to a situation where $g \gg \kappa, \Gamma$. This is realizable in circuit QED or with specific atomic states (e.g Rydberg states) which have a small decay rate.

1. The QuTIP library was developed to help with simulations of quantum problems in Python. Follow the example in the Jupyter notebook to see these vacuum Rabi oscillations, and play around with the parameters : what happens when :

   (a) $\kappa = \Gamma = \Delta_c = \Delta_{at} = 0$? You should recover some standard undamped coherent Rabi oscillations.

(b) $\kappa = \Gamma = 0.1g$?

(c) $\Delta_c = \Delta_{at} = 0$ and $\kappa \ll g < \Gamma$?

(d) $\Delta_c = \Delta_{at} = 0$ and $\Gamma \ll g < \kappa$? Comment the difference with the previous case.

(e) you can also explore around with the detunings Δ_c and Δ_{at}.

2. Relate this to the spectra calculated in section II.

IV. CLASSICAL DESCRIPTION

This section is based on the following article : Tanji-Suzuki, et. al. Interaction between Atomic Ensembles and Optical Resonators, arXiv URL : `https://doi.org/10.48550/arXiv.1104.3594`. Here, we will analyze the atom–light interaction from the classical point of view and derive analytical formulas. We will see that the dimensionless cooperativity parameter $\eta = 4g^2/(\kappa\Gamma)$ governs all aspects of the atom–light interaction both in free-space and in a cavity. The classical derivation for an atom–cavity system agrees with the quantum mechanical result discussed in the previous section.

A. Interaction between a single atom and a free-space mode

In the following, we are interested in the radiation coupled into a mode of interest $\mathcal{M}$ (z-direction) via the atom that is driven by an external field incident from the side (x-direction) with amplitude E. The mode $\mathcal{M}$ is a travelling-wave TEM_{00} Gaussian mode of wavenumber $k = 2\pi/\lambda = \omega/c$ propagating along the z-axis, waist w and Rayleigh range $z_R = \pi w^2/\lambda$. A single atom is located on the optical axis of that mode at the waist. The atom is described as a pointlike classical dipole oscillator. This classical-oscillator description of the atom agrees with the quantum mechanical treatment in the limit where the saturation of the atomic transition is negligible. The external driving field polarization is assumed to be linear and perpendicular to the direction of propagation of the mode $\mathcal{M}$ (y-direction). We would like to know what fraction of the total power scattered by the driven atom is emitted into this specific mode $\mathcal{M}$ versus the power emitted into free-space.

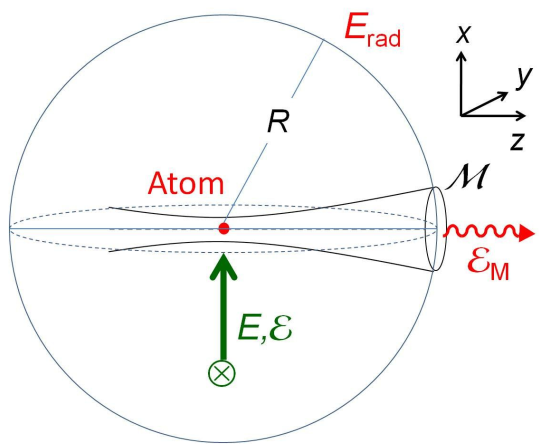

FIG. 1. Scattering of radiation by a weakly driven atom. The incident field $\mathcal{E}$ is polarized perpendicular to the TEM_{00} mode of interest $\mathcal{M}$, which is propagating in the z-direction, (green arrow) and drives an atomic dipole oscillator positioned at the waist of $\mathcal{M}$ (red dot) that emits an electromagnetic field E_{rad} at large distance R from the atom.

The electric field $E_{\mathcal{M}}(\rho, z)$ in mode $\mathcal{M}$ at position (ρ, z) with radial distance $\rho^2 = x^2 + y^2$ from the z-axis can be written as $E_{\mathcal{M}}(\rho, z) = e_{\mathcal{M}}(\rho, z)\mathcal{E}_{\mathcal{M}}/\sqrt{\varepsilon_0 c}$. Here, we will refer to the position-independent quantity $\mathcal{E}_{\mathcal{M}}$ as the mode amplitude and the normalized mode function $e_{\mathcal{M}}$. $\mathcal{E}_{\mathcal{M}}$ is related to the total power $P_{\mathcal{M}}$ via $P_{\mathcal{M}} = |\mathcal{E}_{\mathcal{M}}|^2/2$ and to the electric field at the waist $E_{\mathcal{M}}(0,0)$ via $\mathcal{E}_{\mathcal{M}} = E_{\mathcal{M}}(0,0)\sqrt{\epsilon_0 c A}$, where $A = \pi w^2/2$ is the effective mode area. In the following it will be useful to similarly define a mode amplitude for the driving field E as $\mathcal{E} = \sqrt{\epsilon_0 c A}E$.

At $z \gg z_R$ the normalized mode function $e_{\mathcal{M}}$ is approximated as

$$e_{\mathcal{M}}(\rho, z) \approx \left(\frac{2}{\pi \tilde{w}^2}\right)^{1/2} \exp\left(-\frac{\rho^2}{\tilde{w}^2} + ikz + ik\frac{\rho^2}{2z} - i\frac{\pi}{2}\right), \tag{2}$$

with $\tilde{w}(z) = w\sqrt{1 + (z/z_R)^2}$ the beam spot size.

The driving field E induces an atomic dipole oscillation and the dipole emits a radiation field

$$E_{\rm rad}(R, \theta) = \frac{k^2 \sin\theta}{4\pi\varepsilon_0} \frac{e^{ikR}}{R} \alpha E, \tag{3}$$

at large distance $R \gg \lambda$. Here θ is the angle between the polarization of the driving field and the direction of observation.

The complex polarizability α is given by

$$\alpha = 6\pi\varepsilon_0 c^3 \frac{\Gamma/\omega_0^2}{\omega_0^2 - \omega^2 - i(\omega^3/\omega_0^2)\Gamma}, \tag{4}$$

where ω_0 denotes the atomic resonance frequency and Γ is the linewidth of the atomic transition.

1. First, we derive the mode amplitude $\mathcal{E}_{\mathcal{M}}$ in mode $\mathcal{M}$ arising from the radiated field by calculating the projection $\mathcal{E}_{\mathcal{M}} = \sqrt{\varepsilon_0 c} \int e^*_{\mathcal{M}} E_{\rm rad} 2\pi\rho d\rho$ in the plane $z \gg z_R$.

   Tips:

   - Assume that the mode $\mathcal{M}$ with $w \gg \lambda$ subtends only a small far-field angle $\lambda/(\pi w) \ll 1$, such that the spatial dependence of $E_{\rm rad}$ over the region occupied by $\mathcal{M}$ can be approximated as $\sin\theta \approx 1$ and $e^{ikR}/R \approx e^{ikz + ik\rho^2/(2z)}/z$.
   - To simplify the result, use a dimensionless parameter $\beta = k\alpha/(\pi w^2 \varepsilon_0)$ and a mode amplitude for the driving field $\mathcal{E} = \sqrt{\varepsilon_0 c A} E$ with $A = \pi w^2/2$.

2. **Emission:** The total scattered power into all directions $P_{4\pi}$ can be calculated by integrating the intensity $I_{\rm rad} = \varepsilon_0 c |E_{\rm rad}|^2/2$ of the radiated field over the surface of the sphere of radius R. Derive $P_{4\pi}$ and the ratio of the power emitted into the mode vs. free space $2P_{\mathcal{M}}/P_{4\pi}$ by using the single-atom cooperativity in free space $\eta_{\rm fs} = 6/(k^2 w^2)$.

Now we take the driving field propagating in the z-direction alongside the mode $\mathcal{M}$ with power $P_{\rm in} = |\mathcal{E}|^2/2$. We derive the fraction of the absorbed power from the driving field and the phase shift induced by the atom.

3. Show that within the rotating wave approximation (RWA), $\Delta \equiv \omega - \omega_0 \ll \omega_0$, the mode-coupling parameter β in terms of the light–atom detuning Δ takes the simple form

$$\beta_{\rm RWA} = \eta_{\rm fs}(\mathcal{L}_d(\Delta) + i\mathcal{L}_a(\Delta)), \tag{5}$$

   where $\mathcal{L}_a(\Delta) = \Gamma^2/(\Gamma^2 + 4\Delta^2)$ and $\mathcal{L}_d(\Delta) = -2\Delta\Gamma/(\Gamma^2 + 4\Delta^2)$.

4. **Absorption:** The atomic absorption is derived from the requirement that the power reduction in the forward direction must arise from the destructive interference between the incident field $\mathcal{E}$ and the field $\mathcal{E}_{\mathcal{M}}$ forward-scattered by the atom into the same mode $\mathcal{M}$. Show that in the RWA the ratio of absorbed to incident power $P_{\rm abs}/P_{\rm in}$ is approximated as $2\eta_{\rm fs}\mathcal{L}_a(\Delta)$.

5. **Dispersion:** The driving field in mode $\mathcal{M}$ not only is attenuated, but also experiences a phase shift in the presence of the atom. This phase shift can be understood as arising from the interference of the out-of phase component of the field $\mathcal{E}_{\mathcal{M}}$ forward-scattered by the atom with the incident field $\mathcal{E}$ in the same mode. Derive the atom-induced phase shift ϕ in the RWA.

B. Interaction between a single atom and a cavity mode

Based on the quantitative understanding of atimic emmision into and absorption from a single Gaussian mode in free space we can now analyze the classical interaction between a single atom and a single mode of an optical resonator. Let us consider a standing-wave resonator of length L with two identical, loss-less, partially transmitting mirrors with real amplitude reflection and transmission coefficients r and iq, respectively (r, q real, $r^2 + q^2 = 1$ and $q^2 \ll 1$). We assume that the atom is at rest and ignore light forces and the photon recoil. The resonator supports the mode $\mathcal{M}$, and the atom is located on the mode axis near the waist at an antinode. $\mathcal{E}_{\text{in}}$ is the mode amplitude of the incident field and $\mathcal{E}_c$ the mode amplitude of the travling intracavity field.

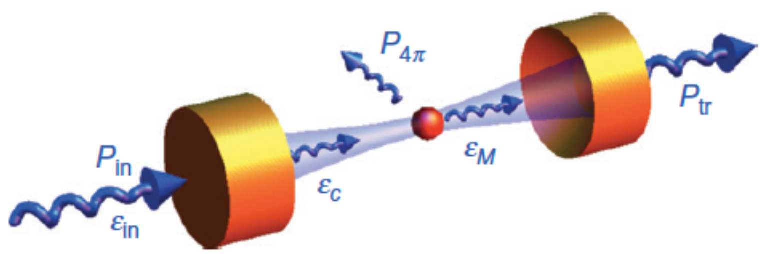

FIG. 2. Transmission through an optical standing-wave resonator containing an atom. An incident field $\mathcal{E}_{in}$ produces a steady-state intracavtiy field with traveling mode amplitude $\mathcal{E}_c$. The atom at an antinode driven by the field $2\mathcal{E}_c$ contributes a field $2\mathcal{E}_{\mathcal{M}}$ per round trip. The transmitted power is P_{tr}, the power scattered by the atom into free space is $P_{4\pi}$.

The steady-state amplitude $\mathcal{E}_c$ can be determined from the condition that the field after one round trip is unchanged:

$$\mathcal{E}_c = r^2 e^{2ikL}\mathcal{E}_c + iq\mathcal{E}_{\text{in}} + 2\mathcal{E}_{\mathcal{M}}. \tag{6}$$

Here the first term is the intracavity mode amplitude after one round-trip acquiring the phase e^{2ikL}. The second is the mode amplitude leaking into the cavity through the input mirror. The third represents the field scattered by the atom with the driving field $2\mathcal{E}_c$, where the factor of 2 arises from simultaneous scattering into both cavity directions by the atom at an antinode.

1. Obtain the ratios of transmitted to incident power $P_{\text{tr}}/P_{\text{in}}$ and scattered to incident power $P_{4\pi}/P_{\text{in}}$, using β, the cavity decay rate $\kappa = q^2 c/L$ and the laser–cavity detuning $\delta \equiv \omega - \omega_c$ with cavity resonance ω_c. For not too large detuning $\delta \ll \pi c/L$ we can approximate $r^2 e^{2ikL} \approx 1 - q^2 + 2iq^2\delta/\kappa$.

2. Rewrite $P_{\text{tr}}/P_{\text{in}}$ and $P_{4\pi}/P_{\text{in}}$ in the RWA using $\mathcal{L}_a(\Delta)$, $\mathcal{L}_d(\Delta)$ and a cavity cooperativity

$$\eta = \frac{4\eta_{\text{fs}}}{q^2} = \frac{24}{q^2 k^2 w^2} = \frac{24\mathcal{F}/\pi}{k^2 w^2}, \tag{7}$$

   with the cavity finesse $\mathcal{F} = \pi c/(L\kappa) = \pi/q^2$. How can the augmentation of the cavity cooperativity η compared to the free-space one η_{fs} be understood?

3. Relate this result to the one of equation (1) in section II.

4. Plot cavity transmission and free-space scattering as a function of Δ in units of Γ for a resonant atom-cavity system ($\omega_c = \omega_0$, i.e. $\delta = \Delta$) in two cases $\kappa = \Gamma$ and $\kappa = 10\Gamma$. For each case plot with the value $\eta = 0.05$ representing the weak-coupling regime ($\eta < 1$) and $\eta = 10$ the strong coupling regime ($\eta > 1$).