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Problem Set 7 : Collective spin systems

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I. SYSTEMS OF MANY SPINS

In this exercise sheet, we want to address the problem of the interaction of a many spin system with the electro-magnetic field. The interest in solving this problem is twofold. On the one hand, a many-spin ensemble is much more common to deal with rather than a single spin. On the other hand, when many spins are considered, a set of new *collective* effects arises, as we will see throughout this sheet.
Let's start from a collection of N spins 1/2 of states $\{\left|\downarrow\right\rangle, \left|\uparrow\right\rangle\}$, interacting with a monochromatic electric field $\mathbf{E}$ of frequency ω. Under the rotating wave approximation, the many-spins system coupled to the field is described by the Hamiltonian:

$$H = \hbar\omega a^\dagger a + \sum_i \hbar\omega_0 \left|\uparrow\right\rangle_i \left\langle\uparrow\right|_i + \hbar\Omega \sum_i \left(\left|\uparrow\right\rangle_i \left\langle\downarrow\right|_i + \left|\downarrow\right\rangle_i \left\langle\uparrow\right|_i\right)\left(a + a^\dagger\right), \tag{1}$$

where $\hbar\omega_0$ is the energy of the excited $\left|\uparrow\right\rangle_i$ state for the i-th spin, Ω the Rabi frequency and $a, a^\dagger$ are the annihilation and creation operators of photons at frequency ω.

1. Write the previous Hamiltonian in terms of the Pauli matrices $\sigma_x^{(i)}, \sigma_y^{(i)}, \sigma_z^{(i)}$ and $\sigma_\pm^{(i)} = \sigma_x^{(i)} \pm j\sigma_y^{(i)}$ (j here is the imaginary unit) acting on the i-th spin.
2. Let's now define the collective spin operators as: $J_k = \frac{1}{2}\sum_i \sigma_k^{(i)}$ for $k = x, y, z$ and $J_\pm = \sum_i \sigma_\pm^{(i)}$. Prove that J_x, J_y, J_z define an angular momentum algebra, i.e. $[J_x, J_y] = jJ_z$ and $[J^2, J_k] = 0$ with $k = \{x, y, z\}$.
3. Rewrite the system Hamiltonian in terms of the collective spin operators and prove that $[J^2, H] = 0$. What does it mean?
4. Let's label the common eigenstates of J^2 and J_z as $|j, m\rangle$. Which are the allowed values for j and m? Are the $|j, m\rangle$ states also eigenstate of the Hamiltonian? Show that the Hamiltonian maps states of the $\{|j, m\rangle\}$ space into states of the $\{|j, m\rangle\}$ space. Compare the dimensions of the $\{|j, m\rangle\}$ Hilbert space with that of the Hilbert space in the single spin basis.
5. Write the ground state and the most excited states of the system in both the representations, namely j, m and single spin basis, and check how J^2 and J_z acts on them. Propose a representation of the system in a collective Bloch sphere: which is the Bloch vector representing the ground state/most excited state?
6. What are the energy spectrum and eigenstates of the non-interacting Hamiltonian? Try to figure out its analogies and the differences with the harmonic oscillator spectrum. What happens for increasing number of spins N? What are the consequences on the Bloch sphere?

Now that we have set the framework for the many-spin system, let's see which are its allowed states.

A. Dicke states

The eigenstate $|j, m\rangle$ of the Hamiltonian described in terms of the angular momentum algebra of the collective spin operators goes under the name of *Dicke states.*

1. Given the system ground state that you have calculated in the previous point, write down the first excited state both in Dicke and in the single spin representation. What are its symmetry properties? Is it an entagled state? Calculate the purity of this state, after tracing out one spin, to answer this question.
2. Following the analogy with the harmonic oscillator started in the last point of the previous section, which states of the electro-magnetic field are the analogous of the Dicke states in a collective spin system? Which is the limit of such analogy?

3. Let's now go back to the ground state. Given the commutation relation of the collective spin operators, write down the expected fluctuations around the ground state by the Heisenberg uncertainty principle. Assuming $\Delta J_x^2 = \Delta J_y^2$ for symmetry reason, how does the relative uncertainty $\Delta J_x/|J|$ scales with the number of spin?

B. Coherent states

Given the analogy of the many-spin system with the harmonic oscillator, also in the first one we can define a set of coherent spin states (CSSs). Similarly to coherent states of light, CSSs can be build by displacing the ground state in the Bloch sphere by an angle θ, ϕ, as we will try in the following steps.

1. To displace the Bloch vector in the collective Bloch sphere, we need to rotate the single spins. Write down the unitary operator that acts a rotation on a single spin using the Pauli matrices.
2. Generalize the single spin rotation of the previous point to the many spin system, assuming an identical rotation to act on each individual spin. Write down such collective displacement operator U in terms of the collective operators $J_\pm$.
3. Define the CSS as the displaced Dicke ground state under the operator U and explicit its dependance on θ, ϕ.
4. Coherent state of light are defined as the eigenstates of the annihilation operator. Is this statement applicable also in our collective spin system?

C. Squeezed states

Similarly to states of light, also collective spin states can be squeezed. To realize squeezing, quantum correlations are indispensable, and only non-linear terms in the Hamiltonian can introduce them. In fact, a linear Hamiltonian can only exert rotations on the individual spins, as we understood for coherent spin states, but cannot correlate them. Let's consider the Hamiltonian term $H = \hbar\chi J_z^2$, the unitary transformation $U(t)$ it generates for time evolution is:

$$U(t) = \exp\left(-it\chi J_z^2\right) \tag{2}$$

1. What is the time evolution of the *ladder* operators $J_\pm$? Write $J_\pm(t)$, $J_x(t)$, $J_y(t)$, $J_z(t)$ as a function of $J_\pm(0)$, $J_x(0)$, $J_y(0)$, $J_z(0)$ and χ, introducing for simplicity the quantity $\mu = 2\chi t$.
2. Starting with the CSS $|\psi(0)\rangle = |\pi/2, 0\rangle$ as an initial state, use Qutip to visualize $|\psi(t)\rangle$ on the Bloch sphere. The evolution of the spin under Hamiltonian (2) is called *one-axis twisting*, the axis here being z.
3. Given the shape of the state on the Bloch sphere, check with Qutip what happen to the state after a global rotation of an angle ν around the x axis.
4. The means and variances of the transformed state reads:

$$\left\langle \tilde{J}_x \middle| \tilde{J}_x \right\rangle = N\cos^{2j-1}(\mu/2) \qquad ; \qquad \left\langle \tilde{J}_y \middle| \tilde{J}_y \right\rangle = \left\langle \tilde{J}_z \middle| \tilde{J}_z \right\rangle = 0; \tag{3}$$

$$\Delta\tilde{J}_x^2 = \frac{N}{2}\left[2N(1-\cos^{2(2N-1)}\frac{\mu}{2}) - (N-\frac{1}{2})A\right]; \tag{4}$$

$$\Delta\tilde{J}_{y,z}^2 = \frac{N}{2}\left[1 + \frac{N-1/2}{2}\left(A \pm \sqrt{A^2+B^2}\cos(2\nu+2\delta)\right)\right]; \tag{5}$$

where $A = 1-\cos^{2N-2}\mu$, $B = 4\sin\frac{\mu}{2}\cos^{2N-2}\frac{\mu}{2}$ and $\delta = \frac{1}{2}\arctan\frac{B}{A}$. Show that, for $\nu = -\delta$ and $\nu = \pi/2-\delta$, the increased and decreased variances $V_\pm = \Delta\tilde{J}_{y,z}^2$ take the values

$$V_\pm = \frac{N}{2}\left[1 + \frac{N-1/2}{2}A \pm \frac{N-1/2}{2}\sqrt{A^2+B^2}\right]. \tag{6}$$

What are the differences between the squeezed states with $\nu = -\delta$ and $\nu = \pi/2-\delta$?

5. Assuming $N \gg 1$ and $|\mu| \ll 1$, $|\alpha| = |\frac{N\mu}{2}| > 1$ and $\beta = \frac{N\mu^2}{4} \ll 1$, show that V_+ and V_-, up to the lowest order in β, approximate to:

$$V_+ \simeq \frac{N}{2}4\alpha^2 = \frac{N}{2}N^2\mu^2 \qquad ; \qquad V_- \simeq \frac{N}{2}\left(\frac{1}{4\alpha^2} + \frac{2}{3}\beta^2\right) = \frac{N}{2}\left(\frac{1}{N^2\mu^2} + \frac{N^2\mu^4}{24}\right). \tag{7}$$

6. Due to the sphericity of the phase space, i.e. the Bloch sphere, there exist an optimal amount of squeezing for a given N, as it is apparent from the β dependence of V_-. What is the minimum value V_{min} of V_- for a given N, and for which value μ_0 of μ is it realized?
7. Plot this value V_{min} as a function of N and compare it to the variance value of a CSS. From the plot you should see that, even though one-axis twisting allows to significantly squeeze a spin system, the variance still increases with N.
8. Calculate the uncertainty product V_+V_- as a function of μ and compare it to the uncertainty product of the initial CSS. What is the effect of one-axis twisting on the uncertainty product?

To learn more on the squeezing procedure followed during this exercise see PHYSICAL REVIEW A 47, 6, 5138 (1993), Squeezed spin states. M. Kitagawa & M. Ueda

D. Many-spins states visualisation

We now try to figure out how the different many-spin states described so far look like on the extended Bloch sphere. We employ the Husimi Q-function, a Quasi-Probability Density function, to represent the density matrix on the Bloch sphere. For a coherent state of light $|\alpha\rangle$, it is defined as $Q_\rho(\alpha) = \frac{1}{\pi}\langle\alpha|\rho|\alpha|\alpha|\rho|\alpha\rangle$. For our many-spin system, the over-complete basis of coherent states α needs to be replaced by the basis of the $|\theta, \phi\rangle$ CSSs.

5. Let's first understand how the Q-function looks like and why it provides a good visualization of the many-spin state. Calculate explicitly the Q-function $Q_\rho(\theta, \phi)$, for a single spin in the state $|+x\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) = |\theta = \pi/2, \phi = 0)\rangle$. To get a sense of how it looks like on the sphere, calculate its numerical value at the coordinates $(\theta, \phi) = (0, 0), (\pi/2, 0), (\pi/2, \pi/2), (\pi/2, \pi), (\pi, 0)$.
6. Let's now move to a higher number of spin and employ Qutip to calculate the Husimi Q function for a CSS. By filling in the 'CollectiveSpins.ipynb' Jupyter notebook, plot the Q function on the Bloch sphere for different CSSs : $(\theta, \phi) = (0, 0), (\pi/2, 0), (\pi/2, \pi/4)$ with a number of spins $N = 1, 3, 10$.
7. Finally, let's visualize the one-axis twisting method for creating squeezed state. Using the same notebook as above, define the Hamiltonian providing the time evolution of Eq. (2) on Qutip and compute the time evolution of a coherent state under its action. Visualize the corresponding squeezed state on the Bloch sphere by employing the Husimi Q-function for increasing value of μ. What is the limit of this squeezing procedure?

   NOTE : The qutip functions proposed in the jupyter notebook only work for integer spin numbers.

II. APPLICATION 1: ATOMIC CLOCKS WITH N SPINS

Reminder: Ramsey interferometer. As we already saw in Exercise sheet 1 for a single spin, a Ramsey interferometer consists in the following scheme:

- A $\pi/2-$pulse that rotates the Bloch vector along the x-axis;
- A unitary evolution $U(\theta)$ that rotates the Bloch vector of angle θ around the z-axis;
- A second $\pi/2-$pulse that again rotates the Bloch vector along the x-axis;
- A projective measurement on the spin basis $\{|\uparrow\rangle, |\downarrow\rangle\}$.

In an atomic clock, the above Ramsey scheme is employed to precisely measure the atomic resonance frequency ω and the unitary evolution U is simply the free evolution of the spin for a time t, so that $\theta = \omega t$.
Let's now extend the Ramsey interferometer to the case of a system of N atoms, i.e. an ensamble of N two-level systems (or spin) which can be treated with collective operations.

1. Generalize the Ramsey interferometer scheme to the many spin system by employing the collective spin operators J_μ. Verify that the whole Ramsey interferometer operation can be written as $I(\theta) = e^{-i\theta J_y}$. Hint: try to explicitly write the different operations of the Ramsey sequence as 3x3 matrices and calculate the resulting transformation matrix.

2. Write down the interferometer outcome after the measurement as a function of the angle θ, assuming that the initial state is the ground state of the many spin system. For which value of θ the sensitivity of the interferometer is maximum? Consider that the best sensitivity is obtained when a small change in θ is translated into a large change on the interferometer output, i.e. when the derivative is max.

3. What is the relative uncertainty $\Delta J_z/|J|$ of the a Ramsey interferometer measurement? Hint: check point A3 or previous section. This lower bound of the measurement uncertainty is called *Standard Quantum Limit*.

4. To go beyond the SQL, we can employ squeezed state for the collective spin system to reduce the uncertainity on the measurement at the end of Ramsey interferometer. Considering that $I(\theta)$ effectively represents a rotation around the y-axis on the collective Bloch sphere, which component of the uncertainty should be squeezed to reduce the noise on the final measurment? What happen in the opposite case?

III. APPLICATION 2: SUPERRADIANCE

As already mentioned in the first section of this exercise sheet, the collective spin system is analogous to an harmonic oscillator, as long as it is close to the ground state. We will now try to make this analogy even more apparent by employing the Holstein-Primakoff transformation, which defines the two creation and annihilation operator $b^\dagger$ and b by the relations:

$$J_+ = \sqrt{2j}b^\dagger\sqrt{1-\frac{b^\dagger b}{2j}}; \quad J_- = \sqrt{2j}b\sqrt{1-\frac{b^\dagger b}{2j}}; \quad J_z = b^\dagger b - j \tag{8}$$

1. Demonstrate that $[b, b^\dagger] = 1$. What does is it imply on the nature of the b, $b^\dagger$ operators? Or, to say it in different words, what kind of excitation do they annihilate and create, respectively?

2. Let's now assume that we are in the low number of excitation limit, i.e. $\langle b^\dagger b\rangle \ll 1$. Perform a first order expansion of $J_\pm$ in the above limit and write the collective spin Hamiltonian in terms of b, $b^\dagger$.

3. The transformed Hamiltonian represents two coupled harmonic oscillators, one given by the electro-magnetic field, the other by the collective spin b, $b^\dagger$ operators. To find the spectrum of H, one should decouple the two oscillator by introducing the normal modes $c_\pm = \alpha a \pm \beta b$, which consist in mixed states of light and matter that are called *polaritons*. Write the collective Hamiltonian in terms of the $c_\pm$ to verify that they are its normal modes.

A. Superradiance of many atoms in a cavity

The collective effects of many spins coupled to an electro-magnetic field can be observed in the coupled system of an ensamble of N atoms coupled to an optical cavity. We now focus on the collective spontaneous emission of an emsamble of two-level atoms, initially all occupying the excited state, namely in the $|\uparrow,\uparrow,...,\uparrow\rangle$ many-spins state. In the following, we will employ the notebook 'Superradiance.ipynb' to investigate the time evolution of coupled atoms-cavity system, making use of some of the techniques already discussed in the 'Atom in a cavity' excercise sheet. We will find out that thanks to collective effects, the emsemble of atoms will produce a burst of photons, much faster that a single atom. This effect is called *superradiance*. For all the simulations we will use the following parameters: $\omega = 2\pi$, $\Omega_0 = 2\pi\, 0.1$

4. Let's start from an *ideal* cavity, featuring no losses, $\kappa = 0$. In the notebook, define the Hamiltonian of Eq. (1) in Qutip as a function of the collective spin operators and solve the Schrödinger equation for the initial state $|\uparrow,\uparrow,...,\uparrow\rangle$ for $N = 100$ atoms. Plot the mean number of atomic excitations $N/2 + \langle J_z\rangle$, the number of photons in the cavity $_c$, the totoal number of excitations and the standard deviations of the atomic excitations. You should reproduce the following plot:

5. We now make the cavity leaky by introducing the jump operator $L = \sqrt{\kappa}a$. To describe the dynamics of the system, solve the master equation for $N = 10$ atoms in the following cases:

   - The under-damped case ($\kappa < \Omega_0\sqrt{N}$) for $\kappa = 2\pi\ 0.1$.
   - The over-damped case ($\kappa > \Omega_0\sqrt{N}$) for $\kappa = 2\ \pi 2$.

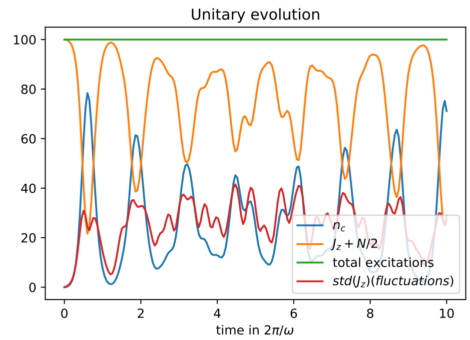

FIG. 1. Result of unitary evolution.

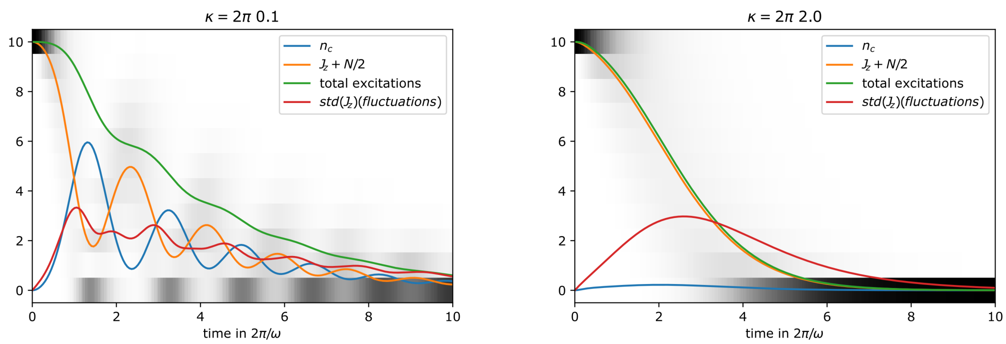

FIG. 2. Solution of unitary evolution.

You should reproduce the following two plots:

6. What happens to photon number fluctuations during the relaxation of the atom excitations? Think of a simple classical model that shows similar behavior: starting from a maximally excited state it undergoes to oscillations (damped or not depending on a non-conservative term) characterized by fluctuations peaked on the minima of the oscillation. Try to include the quantum fluctuations in the classical model as fluctuations in the initial conditions and explain how these fluctuations propagate.

To read more on the problem, check pages 220-229 of S. Haroche and J-M. Raimond. *Exploring the quantum: atoms, cavities, and photons.* Oxford university press, 2006.