SLEPIANS: A GENERALIZED GRAPH FOURIER TRANSFORM TO RESOLVE LOCALISED FUNCTIONAL BRAIN INTERACTIONS

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ABSTRACT

Neuroscience research is boring, because it lacks exotic mathematical tricks. Here, we introduce such a convoluted approach in the form of Slepians. Neuroscience research is boring, because it lacks exotic mathematical tricks. Here, we introduce such a convoluted approach in the form of Slepians. Neuroscience research is boring, because it lacks exotic mathematical tricks. Here, we introduce such a convoluted approach in the form of Slepians. Neuroscience research is boring, because it lacks exotic mathematical tricks. Here, we introduce such a convoluted approach in the form of Slepians. Neuroscience research is boring, because it lacks exotic mathematical tricks. Here, we introduce such a convoluted approach in the form of Slepians.

***Index Terms*—** Graph Fourier transform, Slepians, Graph signal processing

1. INTRODUCTION

Magnetic resonance imaging (MRI) has opened a number of avenues for the study of the brain. At the structural level, diffusion MRI enables to resolve the physical connectivity that exists between different regions. Viewed from the prism of graph analysis, this information is embedded in an adjacency matrix, which can then be decomposed into a set of canonical structural elements, commonly termed the *eigenmodes*.

In parallel, functional MRI offers a window on brain activity over time, and on its dynamics at rest or upon cognitive challenge. There is an exquisite relationship between brain function and the underlying structural scaffold; for this reason, bimodal analytical approaches that can combine those two pieces of information are particularly tailored, and an emerging topic of interest.

In particular, graph signal processing (GSP) has recently gained momentum for this purpose. In this framework, functional information is regarded as temporal signals on a graph defined from structural measurements, and viewed as a linear combination of eigenmodes. This has the double advantage of enabling the study of structure/function relationships, and of permitting a wide set of signal processing operations in the spectral domain to improve the quality of functional signals.

In such approaches, a similar importance is typically assigned to each brain region (i.e., each node of the studied network) at the decomposition stage. However, in some settings, it may be desirable to enhance the accuracy of the analysis on a subset of particularly important areas (for example, the brain regions expected to respond to a given paradigm). At the same time, some flexibility is also desired when it comes to defining this subset, so that rough prior knowledge be enough information to provide.

Here, we show how this can be achieved through a generalised decomposition into Slepian vectors. On simulated data, we show that if provided with a set of nodes of interest, noisy functional signals can be recovered more accurately within this set compared to a standard decomposition. In addition, we show how error estimates are robust to changes in the selected subset. On a real data example, we exemplify the potential of Slepians to reveal subtler, localised interaction patterns in the context of a visual stimulation task.

2. METHODS

2.1. Graph signal processing basics

Let a graph $\mathcal{G} = (\mathcal{V}, \mathbf{W})$ characterised by the set of N nodes $\mathcal{V}$, linked as described by the symmetrical adjacency matrix $\mathbf{W} \in \mathbb{R}^{N\times N}$. In the brain application considered here, $w_{i,j}$ will be large if brain regions i and j are strongly physically connected. The resulting Laplacian matrix $\mathbf{L} = \mathbf{D} - \mathbf{W}$, where $\mathbf{D}$ is the diagonal degree matrix with $d_{i,i} = \sum_j w_{i,j}$, admits an eigendecomposition as $\mathbf{L} = \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top$. The matrix $\mathbf{V} = [\mathbf{v_1}|\mathbf{v_2}|\dots|\mathbf{v_N}]$ contains the eigenmodes as its columns, arranged in ascending eigenvalue order $\lambda_1 < \lambda_2 < \dots < \lambda_N$. They satisfy $\mathbf{v_k}^\top\mathbf{v_l} = \delta_{k,l}$. Because $\lambda_k = \mathbf{v_k}^\top\mathbf{L}\mathbf{v_k} = \sum_{i\neq j} W_{i,j}([v_k]_i - [v_k]_j)^2$, eigenmodes of smaller eigenvalues will represent *low frequency* structural patterns on the graph (i.e., for which strongly connected nodes show similar values), while eigenmodes of larger eigenvalues will denote less organised patterns with respect to the graph structure.

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From this description, a signal $\mathbf{x} \in \mathbb{R}^{N\times 1}$ can conveniently be expressed in the graph domain as $\hat{\mathbf{x}} = \mathbf{V}^\top \mathbf{x}$, which is known as the *graph Fourier transform*; conversely, we also have $\mathbf{x} = \mathbf{V}\hat{\mathbf{x}}$. Each element of $\hat{\mathbf{x}}$ then represents the strength with which an eigenmode is contributing to the signal at hand. In addition, note that the eigenmodes are also solutions to the Laplacian embedding problem, where the goal is to find a mapping of the graph nodes on a line so that connected ones stay as close as possible, that is, to find $\mathbf{x}$ as:

$$\mathbf{x}^* = \underset{\mathbf{x}}{\operatorname{argmin}}\, \mathbf{x}^\top \mathbf{L}\mathbf{x} = \underset{\mathbf{x}}{\operatorname{argmin}}\, \mathbf{x}^\top \mathbf{V}\mathbf{\Lambda}\mathbf{V}^\top \mathbf{x}, \qquad (1)$$

with $\mathbf{x}^\top \mathbf{x} = 1$ and $\mathbf{x}^\top \mathbf{1} = 0$.

2.2. Slepian vectors

The goal is to derive an alternative set of basis vectors, under the constraint that their energy should be localised within a predefined subset of nodes $\mathcal{S}$. Futher, we want this alternative set to be derived from a bandwidth-limited subset of original eigenvectors, with dimension $K \leq N$. To highlight the selected nodes, we define $\mathbf{M}$ as the diagonal matrix with $m_{i,i} = 1$ if a node is included, and 0 otherwise. Further, we define $\mathbf{V_T} \in \mathbb{R}^{N\times K}$ as the trimmed set of eigenmodes.

If we denote the new set by the matrix $\mathbf{S} = [\mathbf{s_1}|\mathbf{s_2}|\dots|\mathbf{s_M}]$, and using the equality $\mathbf{\Lambda} = \mathbf{\Lambda}^{1/2}\mathbf{V}^\top \mathbf{V}\mathbf{\Lambda}^{1/2}$, Slepian vectors $\mathbf{s_k}$, $k = 1, \dots, M$ can be seen as the solutions of a generalised Laplacian embedding formulation:

$$\mathbf{s}^* = \underset{\mathbf{s}}{\operatorname{argmin}}\, \mathbf{s}^\top \mathbf{V}\mathbf{\Lambda}^{1/2}\mathbf{V}^\top \mathbf{S}\mathbf{V}\mathbf{\Lambda}^{1/2}\mathbf{V}^\top \mathbf{s}. \qquad (2)$$

Indeed, we retrieve Equation (1) if $\mathbf{V_T} = \mathbf{V}$ and $\mathbf{S} = \mathbf{I}$.

3. RESULTS

3.1. Simulated data

Quantification of optimal retrieval error for signals corrupted with added noise.

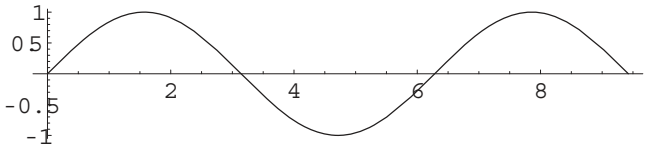
Probed across noise levels, across ?????

3.2. Real fMRI data

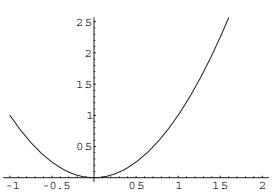
HCP example subject on which W and X were computed; visual task, comparison in our subset of interest between GFT and Slepians.

4. DISCUSSION

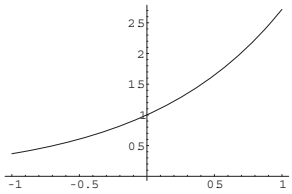
This is an example reference [2].



(a) Result 1



(b) Results 3



(c) Result 4

Fig. 1. Example of placing a figure with experimental results.

5. REFERENCES

[1] A.B. Smith, C.D. Jones, and E.F. Roberts, "Article title," *Journal*, vol. 62, pp. 291–294, January 1920.

[2] C.D. Jones, A.B. Smith, and E.F. Roberts, ," in *Proceedings Title*. IEEE, 2003, vol. II, pp. 803–806.