# Quantum Optics and Quantum Information
# – Spring semester 2023 –
# Solution Set 5 : Entanglement

For questions contact:
Rohit Prasad Bhatt rohit.bhatt@epfl.ch
Francesca Orsi francesca.orsi@epfl.ch
For typos in this exercise solution: Filippo Ferrari filippo.ferrari@epfl.ch

## I. ENTANGLEMENT CARVING.

We consider an ensemble of $N$ spin $1/2$ particles. The states of $1/2$ and $-1/2$ are $|\uparrow\rangle$ and $|\downarrow\rangle$ respectively.

In the cavity, the states of the photon are modeled with a $2D$ Hilbert space spanned by the states $|0\rangle$ if the photon is not detected and $|1\rangle$ if the photon is reflected.

1. We suppose that the initial state of the system is represented by the tensor product $|\psi_S\rangle \otimes |1\rangle$, where $|\psi_S\rangle$ is a generic state of the particles ensemble.

   The described operation can be modeled with an operator $\hat{U}$, acting on the whole system. Let's study first the most general shape of $|\psi_S\rangle$ which distinguishes all the spins pointing up and at least one of the spins pointing up. This will make easier the application of $\hat{U}$ on $|\psi_S\rangle \otimes |1\rangle$. Let's consider the two extrema cases:

   - all the spins are pointing down
   - at least one of the spin is pointing up

   If all the spins are pointing down, then $\left|\psi_S^{(1)}\right\rangle = |\downarrow \ldots \downarrow\rangle$. Otherwise

$$\left|\psi_S^{(2)}\right\rangle = |\psi_S\rangle - \langle\psi_S|\downarrow \ldots \downarrow\rangle \, |\downarrow \ldots \downarrow\rangle \, . \tag{1}$$

   The state $\left|\psi_S^{(2)}\right\rangle$ is orthogonal to $|\downarrow \ldots \downarrow\rangle$, indeed $\left\langle\psi_S^{(2)}\middle|\downarrow \ldots \downarrow\right\rangle = 0$.

   The state $\left|\psi_S^{(2)}\right\rangle$ represents all the possible states with at least one spin up. Now, the state $|\psi_S\rangle$ can be in a generic superposition of $\left|\psi_S^{(1)}\right\rangle$ and $\left|\psi_S^{(2)}\right\rangle$. Therefore we can write

$$|\psi_S\rangle = \alpha \left|\psi_S^{(1)}\right\rangle + \beta \left|\psi_S^{(2)}\right\rangle , \quad \text{with} \ \ |\alpha|^2 + |\beta|^2 = 1. \tag{2}$$

   Finally we apply $\hat{U}$ on $|\psi_S\rangle \otimes |1\rangle$. If all the spins point down, the photon is not detected. If at least one of the spin is pointing up, then the photon is detected. Therefore

$$\begin{aligned} \hat{U}\left(|\psi_S\rangle \otimes |1\rangle\right) &= \alpha \hat{U}\left(\left|\psi_S^{(1)}\right\rangle \otimes |1\rangle\right) + \beta \hat{U}\left(\left|\psi_S^{(2)}\right\rangle \otimes |1\rangle\right) \\ &= \alpha \left|\psi_S^{(1)}\right\rangle \otimes |0\rangle + \beta \left|\psi_S^{(2)}\right\rangle \otimes |1\rangle \, . \end{aligned} \tag{3}$$

   This describes the action of the operator $\hat{U}$ on the whole system $|\psi_S\rangle \otimes |1\rangle$.

2. Now we compute the measurement operators acting on the spin system. The whole system consists in the spin system plus the photon, the last one can be viewed as an environment. The "time evolution" of the whole system is therefore generated by the operator $\hat{U}$. Let's be $\hat{\rho}$ the initial density matrix of the system. The final density matrix must be $\hat{\rho}' = \hat{U}\hat{\rho}\hat{U}^\dagger$.

   Now, the initial density matrix can be recast as $\hat{\rho} = \hat{\rho}_S \otimes |1\rangle\langle 1|$ since we know that the environment is initialized in the state $|1\rangle$.

   Now the goal is to rewrite the evolved density matrix of the spin system in terms of a Kraus Operator Sum Representation (OSR). Indeed, by virtue of Kraus theorem, we know that

$$\hat{\rho}'_S = \sum_\mu \hat{M}_\mu \hat{\rho}_S \hat{M}^\dagger_\mu, \tag{4}$$

   where $\hat{M}_\mu$ are the Kraus meausrement operators. Let's figure out those operators. The evolved density matrix for the system, $\hat{\rho}'_S$ can be obtained tracing out the environment from the total density matrix $\hat{\rho}'$. Then

$$\begin{aligned}\hat{\rho}'_S &= \langle 0|\hat{U}\hat{\rho}\hat{U}^\dagger|0\rangle + \langle 1|\hat{U}\hat{\rho}\hat{U}^\dagger|1\rangle = \langle 0|(\hat{U}\hat{\rho}_S \otimes |1\rangle\langle 1|\hat{U}^\dagger)|0\rangle + \langle 1|(\hat{U}\hat{\rho}_S \otimes |1\rangle\langle 1|\hat{U}^\dagger)|1\rangle \\ &= \langle 0|\hat{U}|1\rangle\hat{\rho}_S(\langle 0|\hat{U}|1\rangle)^\dagger + \langle 1|\hat{U}|1\rangle\hat{\rho}_S(\langle 1|\hat{U}|1\rangle)^\dagger = \hat{M}_0\hat{\rho}_S\hat{M}_0^\dagger + \hat{M}_1\hat{\rho}_S\hat{M}_1^\dagger\end{aligned} \tag{5}$$

   which is the Kraus OSR. Now we write explicitly $\hat{M}_0$ and $\hat{M}_1$.

$$\hat{M}_0 = \langle 0|\hat{U}|1\rangle = |\downarrow ... \downarrow\rangle\langle\downarrow ... \downarrow|, \tag{6}$$

   indeed the action of $\hat{U}$ on $|1\rangle$ generates a superposition of $|0\rangle$ and $|1\rangle$, but only the part proportional to $|0\rangle$ remains after the application of $\langle 0|$. Due to point 1., this part must coincide with $|\downarrow ... \downarrow\rangle\langle\downarrow ... \downarrow|$.

   We have also $\hat{M}_0 + \hat{M}_1 = \mathbb{1}$. Then

$$\hat{M}_1 = \mathbb{1} - \hat{M}_0 = \mathbb{1} - |\downarrow ... \downarrow\rangle\langle\downarrow ... \downarrow|. \tag{7}$$

   The measurements are manifestly projective.

3. Now we consider $|\downarrow ... \downarrow\rangle$ as the initial state of the system. After the global rotation by an angle $\theta$ in $y$ direction the spin are in the final state $|\theta\rangle$.

   The global rotation can be modeled with the operator

$$\hat{R}_y = e^{-i\theta\hat{J}_y}, \quad \text{with} \quad \hat{J}_y = \sum_{i=1}^N \frac{\hat{\sigma}_i^y}{2}. \tag{8}$$

   Therefore

$$|\theta\rangle = e^{-i\theta\hat{J}_y}|\downarrow ... \downarrow\rangle = e^{-i\theta\sum_{i=1}^N \frac{\hat{\sigma}_i^y}{2}}|\downarrow\rangle_1 \otimes ... \otimes |\downarrow\rangle_N = \bigotimes_{i=1}^N e^{-\frac{i}{2}\theta\hat{\sigma}_i^y}|\downarrow\rangle_i, \tag{9}$$

   which is a product state, and therefore not entangled. We can write in a more expressive form $|\theta\rangle$ using the Pauli formula

$$e^{-\frac{i}{2}\theta\hat{\sigma}_i^y} = \cos(\theta/2)\mathbb{1} - i\sin(\theta/2)\sigma_i^y, \tag{10}$$

   obtaining

$$|\theta\rangle = \bigotimes_{i=1}^N [\cos(\theta/2)|\downarrow\rangle_i - \sin(\theta/2)|\uparrow\rangle_i]. \tag{11}$$

   Then, $|\theta\rangle$ is the prduct state of superpositions of spin down and up, modulated by the rotation angle $\theta$.

4. We detect a reflected photon. This means that the state $|\theta\rangle$ evolves under the action of the measurement operator $\hat{M}_1$, i.e.

$$|\phi\rangle = \frac{\hat{M}_1 |\theta\rangle}{\sqrt{\langle\theta| \hat{M}_1^\dagger \hat{M}_1 |\theta\rangle}}. \tag{12}$$

Let's compute first the action of $\hat{M}_1$ on $|\theta\rangle$:

$$\hat{M}_1 |\theta\rangle = |\theta\rangle - \langle\downarrow ... \downarrow|\theta\rangle |\downarrow ... \downarrow\rangle = |\theta\rangle - \left( \bigotimes_{i=1}^{N} \langle\downarrow|_i \left[\cos(\theta/2) |\downarrow\rangle_i - \sin(\theta/2) |\uparrow\rangle_i\right] \right) |\downarrow ... \downarrow\rangle . \tag{13}$$

Therefore

$$\hat{M}_1 |\theta\rangle = |\theta\rangle - \cos^N(\theta/2) |\downarrow ... \downarrow\rangle , \quad \text{and} \tag{14}$$

$$\langle\theta| \hat{M}_1^\dagger \hat{M}_1 |\theta\rangle = \left(\langle\theta| - \cos^N(\theta/2) \langle\downarrow ... \downarrow|\right) \left(|\theta\rangle - \cos^N(\theta/2) |\downarrow ... \downarrow\rangle\right) = 1 - \cos^{2N}(\theta/2), \tag{15}$$

and we finally get the explicit expression for $|\phi\rangle$ in terms of $|\theta\rangle$ and $|\downarrow ... \downarrow\rangle$

$$|\phi\rangle = \frac{|\theta\rangle - \cos^N(\theta/2) |\downarrow ... \downarrow\rangle}{\sqrt{1 - \cos^{2N}(\theta/2)}}. \tag{16}$$

5. A many-body quantum state is said to be entangled if it can not be expressed as a tensor product of one-body quantum states (a product state). A popular "measurement" of entanglement consists in calculating the purity of the reduced density matrix $\hat{\rho}_{\text{red}}$, $\gamma = \text{Tr}(\hat{\rho}^2_{\text{red}})$. If $\gamma = 1$, then the reduced density matrix was pure, and the state was not entangled. If instead $\gamma < 1$, then the reduced density matrix was mixed, and the state was entangled.

   Let's consider the many-body state

$$|W\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |\downarrow ... \uparrow_i ... \downarrow\rangle , \tag{17}$$

   let's prove that this is an entangled state computing the purity of the obtained reduced density matrix after have traced out one of the spins.

   The simplest case is $N = 2$ and it is propaedeutic to the general $N$ spins system. For $N = 2$ we get

$$|W\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle), \tag{18}$$

   which is a Bell state. The associated density matrix is obviously $\hat{\rho} = |W\rangle \langle W|$. The reduced density matrix is given by

$$\hat{\rho}_{\text{red}} = \text{Tr}_1(\hat{\rho}) = \langle\downarrow|_1 |W\rangle \langle W|\downarrow\rangle_1 + \langle\uparrow|_1 |W\rangle \langle W|\uparrow\rangle_1 = \frac{1}{2}(|\downarrow\rangle \langle\downarrow| + |\uparrow\rangle \langle\uparrow|), \tag{19}$$

   which is mixed. Then the purity is

$$\gamma = \text{Tr}(\hat{\rho}^2_{\text{red}}) = \frac{1}{2} < 1, \tag{20}$$

   and the initial state was entangled. We have verified that Bell states are actually entangled.

Now let's prove that even the $N$ spins case is entangled. Let's write

$$|W\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}|\downarrow ... \uparrow_i ... \downarrow\rangle = \frac{1}{\sqrt{N}}|\uparrow\rangle_1 \otimes |\downarrow ... \downarrow\rangle_{N-1} + \frac{1}{\sqrt{N}}|\downarrow\rangle_1 \otimes \sum_{i=1}^{N-1}|\downarrow ... \uparrow_i ... \downarrow\rangle . \tag{21}$$

Now the associated density matrix can be readily constructed as $\hat{\rho} = |W\rangle\langle W|$, while the reduced density matrix is, as in the $N = 2$, $\hat{\rho}_{\text{red}} = \langle\downarrow|_1 |W\rangle\langle W|\downarrow\rangle_1 + \langle\uparrow|_1 |W\rangle\langle W|\uparrow\rangle_1$. One obtain easily

$$\hat{\rho}_{\text{red}} = \frac{1}{N}|\downarrow ... \downarrow\rangle\langle\downarrow ... \downarrow|_{N-1} + \frac{1}{N}\sum_{i,j=1}^{N-1}|\downarrow ... \uparrow_i ... \downarrow\rangle\langle\downarrow ... \uparrow_j ... \downarrow|_{N-1}\,, \tag{22}$$

and the squared density matrix is given by

$$\begin{aligned}\hat{\rho}^2_{\text{red}} = \frac{1}{N^2}|\downarrow ... \downarrow\rangle\langle\downarrow ... \downarrow|_{N-1} + \frac{1}{N^2}\left(\sum_{i,j=1}^{N-1}|\downarrow ... \uparrow_i ... \downarrow\rangle\langle\downarrow ... \uparrow_j ... \downarrow|_{N-1}\right)\\ \left(\sum_{k,l=1}^{N-1}|\downarrow ... \uparrow_k ... \downarrow\rangle\langle\downarrow ... \uparrow_l ... \downarrow|_{N-1}\right) = \frac{1}{N^2}|\downarrow ... \downarrow\rangle\langle\downarrow ... \downarrow|_{N-1} + \frac{1}{N^2}\sum_{i,j,k,l}^{N-1}\delta_{kl}|\downarrow ... \uparrow_i ... \downarrow\rangle\\ \langle\downarrow ... \uparrow_l ... \downarrow|_{N-1} = \frac{1}{N^2}|\downarrow ... \downarrow\rangle\langle\downarrow ... \downarrow|_{N-1} + \frac{N-1}{N^2}\sum_{i,l}^{N-1}|\downarrow ... \uparrow_i ... \downarrow\rangle\langle\downarrow ... \uparrow_l ... \downarrow|_{N-1}\,.\end{aligned} \tag{23}$$

The purity is finally

$$\gamma = \text{Tr}(\hat{\rho}^2_{\text{red}}) = \frac{1}{N^2} + \frac{(N-1)^2}{N^2} = \frac{N^2 - 2N + 2}{N^2} < 1. \tag{24}$$

Then the state $|W\rangle$ was entangled. Notice that for large $N$, $\gamma \sim 1 - 2/N$.

6. We now want to understand if our measurement protocol is able to construct the state $|W\rangle$. In order to answer to the question, we compute the fidelity $\mathcal{F}(W, \phi) = |\langle W|\phi\rangle|^2$ for $\theta \ll 1$.
Let's compute first the overlap $\langle W|\phi\rangle$.

$$\langle W|\phi\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\langle\downarrow ... \uparrow_i ... \downarrow|\phi\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{\langle\downarrow ... \uparrow_i ... \downarrow|\theta\rangle}{\sqrt{1-\cos^{2N}(\theta/2)}}, \tag{25}$$

because $\langle\downarrow ... \uparrow_i ... \downarrow|\downarrow ... \downarrow\rangle = 0$. The overlap $\langle\downarrow ... \uparrow_i ... \downarrow|\theta\rangle$ is given by

$$\begin{aligned}\langle\downarrow ... \uparrow_i ... \downarrow|\bigotimes_{j=1}^{N}[\cos(\theta/2)|\downarrow\rangle_j - \sin(\theta/2)|\uparrow\rangle_j] = (\langle\downarrow|_1[\cos(\theta/2)|\downarrow\rangle_1 - \sin(\theta/2)|\uparrow\rangle_1])\ ...\\ (\langle\uparrow|_i[\cos(\theta/2)|\downarrow\rangle_i - \sin(\theta/2)|\uparrow\rangle_i])\ ...\ (\langle\downarrow|_N[\cos(\theta/2)|\downarrow\rangle_N - \sin(\theta/2)|\uparrow\rangle_N]) = -\cos^{N-1}(\theta/2)\sin(\theta/2).\end{aligned} \tag{26}$$

Then we obtain

$$\langle W|\phi\rangle = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{-\cos^{N-1}(\theta/2)\sin(\theta/2)}{\sqrt{1-\cos^{2N}(\theta/2)}} = -\sqrt{N}\frac{\cos^{N-1}(\theta/2)\sin(\theta/2)}{\sqrt{1-\cos^{2N}(\theta/2)}}, \tag{27}$$

and the fidelity is given by

$$\mathcal{F}(W, \phi) = N\frac{\cos^{2N-2}(\theta/2)\sin^2(\theta/2)}{1-\cos^{2N}(\theta/2)}. \tag{28}$$

Now we study the limit $\theta \to 0$.

$$\mathcal{F} \simeq N \frac{\left(1-\frac{\theta^2}{8}\right)^{2N-2} \frac{\theta^2}{4}}{1-\left(1-\frac{\theta^2}{8}\right)^{2N}} \simeq N \frac{\left[1-(2N-2)\frac{\theta^2}{8}\right] \frac{\theta^2}{4}}{1-1+2N\frac{\theta^2}{8}} = 1. \tag{29}$$

7. We demonstrated that the proposed protocol can be used for generating entangled states starting from the product state $|{\downarrow} \ldots {\downarrow}\rangle$.

   A possible limitation of such a protocol can be the need of a high signal/noise ratio for measuring photons.

## II. PERMUTATIONS AND ENTANGLEMENT MEASUREMENTS.

We consider a system comprising two identical ensembles of qubits, let $\hat{V}$ the SWAP operator that exchanges the states between the two ensembles

$$\hat{V} \left|\phi\right\rangle_1 \otimes \left|\psi\right\rangle_2 = \left|\psi\right\rangle_1 \otimes \left|\phi\right\rangle_2 , \tag{30}$$

where $|\phi\rangle$ and $|\psi\rangle$ are generic states in the Hilbert space of one of the two ensembles (of course $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$).

1. Let's first prove that the eigenvalues of $\hat{V}$ are $\pm 1$. We apply $\hat{V}^2$ to $\left|\phi\right\rangle_1 \otimes \left|\psi\right\rangle_2$ obtaining

$$\hat{V}^2 \left|\phi\right\rangle_1 \otimes \left|\psi\right\rangle_2 = \hat{V} \left|\psi\right\rangle_1 \otimes \left|\phi\right\rangle_2 = \left|\phi\right\rangle_1 \otimes \left|\psi\right\rangle_2 . \tag{31}$$

   Then $\hat{V}^2 = \mathbb{1}$ and the eigenvalues of $\hat{V}$ are $\pm 1$.

2. We prepare the ensembles such that both are identical copies of each other $\hat{\rho}_{\text{full}} = \hat{\rho} \otimes \hat{\rho}$. First, we want to show that $\langle \hat{V} \rangle_{\text{full}} = \text{Tr}(\hat{V}\hat{\rho}_{\text{full}}) = \text{Tr}(\hat{\rho}^2) = \gamma$. In other words, the expectation value of the operator $\hat{V}$ coincides with the purity $\gamma$ of the state of an individual ensemble.

   The most general form of $\hat{\rho}$ is by definition

$$\hat{\rho} = \sum_i p_i \left|\phi_i\right\rangle \left\langle\phi_i\right| , \quad \text{and} \tag{32}$$

$$\hat{\rho} \otimes \hat{\rho} = \sum_i p_i \left|\phi_i\right\rangle \left\langle\phi_i\right| \otimes \sum_j p_j \left|\phi_j\right\rangle \left\langle\phi_j\right| = \sum_{i,j} p_i p_j \left|\phi_i \phi_j\right\rangle \left\langle\phi_i \phi_j\right| . \tag{33}$$

   Let's apply explicitly $\hat{V}$ to $\hat{\rho} \otimes \hat{\rho}$.

$$\hat{V} \hat{\rho} \otimes \hat{\rho} = \hat{V} \sum_{i,j} p_i p_j \left|\phi_i \phi_j\right\rangle \left\langle\phi_i \phi_j\right| = \sum_{i,j} p_i p_j \left|\phi_j \phi_i\right\rangle \left\langle\phi_i \phi_j\right| . \tag{34}$$

   Now we take the trace over the whole system

$$\begin{aligned} \text{Tr}(\hat{V}\hat{\rho}_{\text{full}}) &= \sum_{k,l} \left\langle\phi_k \phi_l\right| \left( \sum_{i,j} p_i p_j \left|\phi_j \phi_i\right\rangle \left\langle\phi_i \phi_j\right| \right) \left|\phi_k \phi_l\right\rangle \\ &= \sum_{i,j,k,l} \delta_{kl} \delta_{il} p_i p_j = \sum_i p_i^2 . \end{aligned} \tag{35}$$

   Now we compute $\hat{\rho}^2$.

$$\hat{\rho}^2 = \left(\sum_i p_i |\phi_i\rangle \langle\phi_i|\right)\left(\sum_j p_j |\phi_j\rangle \langle\phi_j|\right) = \sum_i p_i^2 |\phi_i\rangle \langle\phi_i|, \tag{36}$$

and the trace is trivially $\sum_i p_i^2$.

In order to show that if $\hat{\rho}$ is pure, then $\hat{\rho}_{\text{full}}$ is symmetric, we can proceed in two ways.
First, we can apply the definition of pure state. If $\hat{\rho}$ is pure, then $\hat{\rho} = |\phi\rangle\langle\phi|$, and therefore $\hat{\rho}_{\text{full}} = \hat{\rho}\otimes\hat{\rho} = |\phi\phi\rangle\langle\phi\phi|$, and we get

$$\hat{V}|\phi\phi\rangle\langle\phi\phi| = |\phi\phi\rangle\langle\phi\phi|, \tag{37}$$

this means that $|\phi\phi\rangle\langle\phi\phi|$ is an eigenstate of $\hat{V}$ with eigenvalues $+1$, so it is symmetric.
Second, more profoundly, we can write down the most general form of $\hat{V}$ distinguishing the symmetric and antisymmetric part

$$\hat{V} = (+1)\sum_{\text{sym}|\xi\rangle} |\xi\rangle\langle\xi| + (-1)\sum_{\text{antisym}|\zeta\rangle} |\zeta\rangle\langle\zeta|. \tag{38}$$

Now, if $\hat{\rho}$ is pure, then $\text{Tr}(\hat{V}\hat{\rho}_{\text{full}}) = \text{Tr}(\hat{\rho}^2) = 1$. Therefore, if $\text{Tr}(\hat{V}\hat{\rho}_{\text{full}}) = 1$ (i.e., $\hat{\rho}$ is pure) then $\hat{\rho}_{\text{full}}$ must contain only symmetric wavefunctions, by virtue of the general form of $\hat{V}$.
This second argument, although less immediate, allows to deduce symmetry from purity directly from the value of $\text{Tr}(\hat{V}\hat{\rho}_{\text{full}})$.

3. The fundamental problem in the preparation of $\hat{\rho}_{\text{full}}$ is the difficulty of cloning quantum states. In general, given a state described by the density matrix $\hat{\rho}$, we can not make a copy of it.

   A possible solution is the initialization of two identical $\hat{\rho}$ instead of the copy of it.

4. We now focus on the simplest scenario where each ensemble contains one qubit. The goal is measuring the purity of quantum states upon the preparation of two identical copies of the same system. Let's write first the basis states of the symmetric and antisymmetric subspaces of the Hilbert space in the $\{|\uparrow\rangle, |\downarrow\rangle\}$ basis.

   We have simply the symmetric triplet

$$\left\{|\uparrow\uparrow\rangle, |\downarrow\downarrow\rangle, \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)\right\}, \tag{39}$$

   and the antisymmetric singlet

$$\left\{\frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)\right\}. \tag{40}$$

5. The density matrix of the one-qubit system is

$$\hat{\rho} = p_\uparrow |\uparrow\rangle\langle\uparrow| + p_\downarrow |\downarrow\rangle\langle\downarrow|. \tag{41}$$

   The whole system is $\hat{\rho}_{\text{full}} = \hat{\rho}\otimes\hat{\rho}$. From the previous point we know the $4D$ basis of the full Hilbert space, then we can write the density matrix of the two-qubits system $\hat{\rho}_{\text{full}}$ in the new basis.
   Let's call for simplicity $|1\rangle = |\uparrow\uparrow\rangle$, $|2\rangle = |\downarrow\downarrow\rangle$, $|3\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle) = |\Psi^+\rangle$ and $|4\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle) = |\Psi^-\rangle$. The full density matrix is therefore

$$\hat{\rho}_{\text{full}} = \sum_{i=1}^{4} p_i |i\rangle\langle i|. \tag{42}$$

Now we apply the SWAP operator

$$\hat{V}\hat{\rho}_{\text{full}} = \sum_{i=1}^{3} p_i \left|i\right\rangle \left\langle i\right| - p_4 \left|4\right\rangle \left\langle 4\right| = \sum_{i=1}^{3} p_i \left|i\right\rangle \left\langle i\right| - p_- \left|\Psi^-\right\rangle \left\langle \Psi^-\right|, \tag{43}$$

since $|4\rangle = |\Psi^-\rangle$ is the only antisymmetric state. Now we compute the trace

$$\text{Tr}(\hat{V}\hat{\rho}_{\text{full}}) = \text{Tr}\left(\sum_{i=1}^{3} p_i \left|i\right\rangle \left\langle i\right| - p_- \left|\Psi^-\right\rangle \left\langle \Psi^-\right|\right) = \sum_{i=1}^{3} p_i - p_- = \sum_{i=1}^{4} p_i - 2p_-. \tag{44}$$

But $\sum_{i=1}^{4} p_i = 1$ since they are probabilities and we get $\text{Tr}(\hat{V}\hat{\rho}_{\text{full}}) = 1 - 2p_-$. For point 2., we know that $\text{Tr}(\hat{V}\hat{\rho}_{\text{full}}) = \text{Tr}(\hat{\rho}^2) = \gamma$ and we conclude

$$p_- = \frac{1-\gamma}{2}, \tag{45}$$

which is the probability to measure the two qubits in the Bell state $|\Psi^-\rangle$.